SPRING 2023 MATH 590: EXAM I: SOLUTIONS AND COMMENTS

Name:

Throughout V will denote a vector space over $F = \mathbb{R}$ or $\mathbb{C}$.

(I) **True-False:** Write true or false next to each of the statements below. (3 points each)

(a) $\mathbb{R}^{17}$ can be spanned by 19 vectors. True

Comment. If $B := \{e_1, \dots, e_{17}\}$ is the standard basis for $\mathbb{R}^{17}$, adding two more vectors to this set will still span $\mathbb{R}^{17}$ because the space cannot get any bigger.

(b) The only proper subspaces of $\mathbb{R}^3$ are planes through the origin. False

Comment. Lines through the origin also give proper subspaces.

(c) Ten linearly independent vectors in $\mathbb{R}^{10}$ form a basis for $\mathbb{R}^{10}$. True

Comment. That n linearly independent vectors form a basis for a vector space of dimension n was mentioned several times in class, and given as a corollary to a theorem. See the Daily Update of February 3.

(d) If $v_1, \dots, v_r \in V$ are linearly independent and $v_{r+1} \in V$, then $v_1, \dots, v_{r+1}$ are linearly independent. False

Comment. If v_{r+1} is in $\text{Span}\{v_1, \dots, v_r\}$, then the vectors $v_1, \dots, v_r, v_{r+1}$ are linearly dependent.

(e) Suppose $V = \text{Span}\{v_1, v_2, v_3, v_4\}$ and $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = \vec{0}$, with each $a_i \in F$ and $a_1 \neq 0$. Then $V = \text{Span}\{v_2, v_3, v_4\}$. True

Comment. Since $a_1 \neq 0$, the equation $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = \vec{0}$ can be solved for v_1 in terms of v_2v_3, v_4, showing that v_1 is a redundant vector - thus upon deleting v_1, the space spanned by the remaining vectors stays the same.

(II) State the indicated definition, proposition or theorem. (5 points each)

(a) Given independent vectors $u_1, \dots, u_r \in V$ and vectors $w_1, \dots, w_t \in V$ such that $\text{Span}\{w_1, \dots, w_t\} = V$, state the conclusion of ExchangeTheorem as it applies in this case.

Solution. We have $r \leq t$ and after re-indexing the w_i, $V = \text{Span}\{u_1, \dots, u_r, w_{r+1}, \dots, w_t\}$.

Comment. Both statements are important components of the conclusion of the Exchange Property.

(b) Given $\alpha = \{v_1, \dots, v_n\}$ a basis for V and $\beta = \{w_1, \dots, w_m\}$ a basis for W, define $[T]_\alpha^\beta$, for $T : V \to W$ a linear transformation.

Solution. For each $1 \leq j \leq n$, write $T(v_j) = a_{1j}w_1 + \cdots + a_{mj}w_m$, with each $a_{ij} \in F$. Then $[T]_\alpha^\beta$ is the $m \times n$ matrix whose (i, j)th entry is a_{ij}. Equivalently, $[T]_\alpha^\beta$ is the $m \times n$ matrix whose jth column is $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

Comment. This definition was given multiple times in class, appears on Quiz 4 in exactly the same form - (with its solution posted), and in the Daily Update of February 8.

(c) Suppose $T : V \to W$ is a linear transformation, α_1, α_2 are bases for V and β_1, β_2 are bases for W. State the General Change of Basis Theorem relating the matrix of T with respect to α_1 and β_1 to the matrix of T with respect to α_2 and β_2.

Solution. $[T]_{\alpha_2}^{\beta_2} = [I]_{\beta_1}^{\beta_2} \cdot [T]_{\alpha_1}^{\beta_1} \cdot [I]_{\alpha_2}^{\alpha_1}$.

Comment. See the General Change of Basis Theorem in the Daily Update of February 15.

(III) Short Answer. (15 points each)

(a) Suppose $v_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 5 \end{pmatrix}$. Write a matrix equation you would solve to determine if the vector $w = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ belongs to the span of v_1, v_2, v_3 and **explain what a possible solution to this equation means**. Do not work out the details of solving the matrix equation.

Solution. $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -4 \\ 0 & 1 & 0 \\ -2 & 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$. The vector w belongs to $\text{Span}\{v_1, v_2, v_3\}$ if this equation has a solution. If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a solution, then $w = av_1 + bv_2 + cv_3$.

Comment. See the Daily Update of January 25 which gives exactly these statements for the general case of determining when when a vector w is in the span of vectors $v_1, \ldots, v_n$.

(b) Consider the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x, y) = (2x - y, x - 2y, x + y)$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $S(x, y, z) = (x - y + z, 4x + 3y + 2z)$. **State and verify** the formula that expresses the matrix of ST in terms of the matrices for S and T, using the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution. If we write $E = \{e_1, e_2\}$ for the standard basis of $\mathbb{R}^2$ and $F = \{f_1, f_2, f_3\}$ for the standard basis of $\mathbb{R}^3$, then $[ST]_E^E = [S]_F^E \cdot [T]_E^F$.

1. To calculate $[ST]_E^E$:
   (i) $ST(e_1) = S(T(e_1)) = S(2, 1, 1) = (2, 13) = 2 \cdot e_1 + 13 \cdot e_2$
   (ii) $ST(e_2) = S(T(e_2)) = S(-1, -2, 1) = (2, -8) = 2 \cdot e_1 + -8 \cdot e_2$.

Thus, $[ST]_E^E = \begin{pmatrix} 2 & 2 \\ 13 & -8 \end{pmatrix}$.

2. To calculate: $[T]_E^F$: $T(e_1) = (2, 1, 1) = 2 \cdot f_1 + 1 \cdot f_2 + 1 \cdot f_3$ and $T(e_2) = (-1, -2, 1) = -1 \cdot f_1 + -2 \cdot f_2 + 1 \cdot f_3$.

Thus $[T]_E^F = \begin{pmatrix} 2 & -1 \\ 1 & -2 \\ 1 & 1 \end{pmatrix}$.

3. To calculate $[S]_F^E$: $S(f_1) = S(1, 0, 0) = (1, 4) = 1 \cdot e_1 + 4 \cdot e_2$, $S(f_2) = S(0, 1, 0) = (-1, 3) = -1 \cdot e_1 + 3 \cdot e_2$, $S(f_3) = (0, 0, 1) = (1, 2) = 1 \cdot e - 1 + 2 \cdot e_2$.

Thus, $[S]_F^E = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 3 & 2 \end{pmatrix}$.

Thus,

$$[S]_F^E \cdot [T]_E^F = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 13 & -8 \end{pmatrix} = [ST]_E^E.$$

Comment. A homework problem of exactly of this type (with more difficult bases, but dimension two spaces) was given on February 10.

(c) Suppose α and β are bases for V. Let P be the matrix obtained by writing the vectors in α in terms of the vectors in β. Describe in terms of P the matrix obtained by writing the vectors in β in terms of the vectors in α. Justify your answer.

Solution. Note that $P = [I]_\alpha^\beta$. Then writing the basis β in terms of α gives $[I]_\beta^\alpha$, which is P^{-1}, since

$$[I]_\alpha^\beta \cdot [I]_\beta^\alpha = [I]_\alpha^\alpha = \text{ identity matrix } = [I]_\beta^\beta = [I]_\beta^\alpha \cdot [I]_\alpha^\beta.$$

Comment. See the discussion concerning the Change of Basis Theorem in the Daily Update of February 13 and the product formula presented on February 10.

(IV) State and prove the Rank plus Nullity Theorem (called the Dimension Theorem in our textbook.) (25 points)

Solution. **Rank plus nullity theorem.** Let $T : V \to W$ be a linear transformation between finite dimensional vectors spaces. Then:

$$\dim(V) = \dim(\text{im}(T)) + \dim(\ker(T)).$$

Proof. Set $r := \dim(\ker(T))$ and $n := \dim(V)$. Let $v_1, \ldots, v_r$ be a basis for $\dim(\ker(T))$. We may extend this set to a basis $v_1, \ldots, v_r, v_{r+1}, \ldots, v_n$ for V. We claim $B := \{T(v_{r+1}), \ldots, T(v_n)\}$ is a basis for $\text{im}(T)$. If the claim holds, then

$$\dim(V) = n = (n - r) + r = \dim(\text{im}(T)) + \dim(\ker(T)),$$

which proves the theorem.

B spans $\text{im}(T)$. Take $w \in \text{im}(T)$. Then $w = T(v)$ for some $v \in V$. We may write

$$v = a_1v_1 + \cdots + a_rv_r + a_{r+1}v_{r+1} + \cdots a_nv_n,$$

since $v_1, \ldots, v_n$ is a basis for V. Applying T to both sides of this equation, we get

$$\begin{aligned} w = T(v) &= a_1T(v_1) + \cdots + a_rT(v_r) + a_{r+1}T(v_{r+1}) + \cdots + a_nT(v_n) \\ &= \vec{0} + + \cdots \vec{0} + a_{r+1}T(v_r) + \cdots + a_nT(v_n) \\ &= a_{r+1}T(v_r) + \cdots + a_nT(v_n), \end{aligned}$$

since each $v_i \in \ker(T)$, for $1 \le i \le r$. This shows B spans $\text{im}(T)$.

Linear independence of B. Suppose $b_{r+1}T(v_{r+1}) + \cdots + b_nT(v_n) = \vec{0}$, with each $b_i \in F$. Since T is a linear transformation, we have $T(b_{r_1}v_{r+1} + \cdots + b_nv_n) = \vec{0}$, showing that $b_{r+1}v_{r+1} + \cdots + b_nv_n \in \ker(T)$. Writing this last vector in terms of the basis for $\ker(T)$, we have

$$b_{r+1}v_{r+1} + \cdots + b_nv_n = b_1v_1 + \cdots + b_rv_r,$$

for $b_1, \ldots, b_r \in F$. Thus,

$$(*) \qquad -b_1v_1 - \cdots - b_rv_r + b_{r+1}v_{r+1} + \cdots + b_nv_n = \vec{0},$$

in V. Since $v_1, \ldots, v_n$ form a basis for V, all of the coefficients in (*) are 0. In particular, $b_{r+1} = \cdots = b_n = 0$, showing that the vectors in B are linearly independent. Hence B is a basis for $\text{im}(T)$, completing the proof of the claim, and the proof of the theorem.

Comment. The statement and proof were presented in class February 20. When presenting a theorem, it is important to present and state clearly all premises of various parts of the argument. A proof is more than a sequence of equations, but requires appropriate exposition as well.